

CENTRALIZERS OF RANK-1 HOMEOMORPHISMS

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ABSTRACT. We give a definition for a rank-1 homeomorphism of a zero-dimensional Polish space X . We show that if a rank-1 homeomorphism of X satisfies a certain non-degeneracy condition, then it has trivial centralizer in the group of all homeomorphisms of X , i.e., it commutes only with its integral powers.

1. INTRODUCTION

Let $\text{Aut}(X, \mu)$ denote the group of invertible measure-preserving transformations of a standard Lebesgue space (X, μ) , taken modulo null sets and equipped with the weak topology. The subset of $\text{Aut}(X, \mu)$ consisting of rank-1 transformations has been extensively studied. One of the important results about rank-1 transformations is King's weak closure theorem [6]:

Theorem 1.1 (King, 1986). *If $T \in \text{Aut}(X, \mu)$ is rank-1, then the centralizer of T in the group $\text{Aut}(X, \mu)$ equals $\overline{\{T^i : i \in \mathbb{Z}\}}$.*

Some rank-1 measure-preserving transformations commute only with their integral powers (e.g., Chacon's transformation, see [5]), but this is not typical. It is well known that a generic measure-preserving transformation T is rigid, and thus $\overline{\{T^i : i \in \mathbb{Z}\}}$ is uncountable. Stepin and Ereminko showed in [2] that every compact abelian group embeds into the centralizer of a generic measure-preserving transformation. Glasner and Weiss showed in [4] that for generic T , there is no spatial realization of $\overline{\{T^i : i \in \mathbb{Z}\}}$, which, by a theorem of Mackey, implies that $\overline{\{T^i : i \in \mathbb{Z}\}}$ is not locally compact. Even more recently, Mellerey and Tsankov showed that for generic T , the group $\overline{\{T^i : i \in \mathbb{Z}\}}$ is extremely amenable. As a generic measure-preserving transformation is rank-1,

these results imply, in a very strong way, that for rank-1 transformations, $\overline{\{T^i : i \in \mathbb{Z}\}}$ typically contains much more than $\{T^i : i \in \mathbb{Z}\}$.

In this paper we introduce the notion of a rank-1 homeomorphism of a zero-dimensional Polish space X . Our main result is that if such a rank-1 homeomorphism f satisfies a certain non-degeneracy condition, it has trivial centralizer in the group of homeomorphisms of X , i.e., the centralizer of f in the group $\text{Homeo}(X)$ equals $\{f^i : i \in \mathbb{Z}\}$.

To motivate this, we will first describe how homeomorphisms can naturally be obtained from rank-1 measure-preserving transformations. In the literature, there are several different definitions of rank-1 transformations; these are all equivalent if we restrict our attention to transformations that are totally ergodic, i.e. those T for which T^n is ergodic for each $n > 0$. A nice discussion of these several definitions and their interconnections can be found in the survey article [3]. Below we mention two ways of defining rank-1 transformations and how one can realize rank-1 measure-preserving transformations as homeomorphisms of zero-dimensional Polish spaces.

One way of defining rank-1 transformations is via symbolic systems. One first defines a collection of symbolic rank-1 systems. Each such system is a triple (X, μ, σ) , where X is some closed, shift-invariant subset of $2^{\mathbb{Z}}$ with no isolated points, μ is a shift-invariant, non-atomic probability measure supported on X , and σ is the shift. A totally ergodic measure-preserving transformation is rank-1 if it is isomorphic to some symbolic rank-1 system. For a symbolic rank-1 system, the shift σ is not just a measure-preserving transformation, but also a homeomorphism of the Cantor space X . King's theorem gives us information about the centralizer of σ in the group $\text{Aut}(X, \mu)$, but one can also ask about the centralizer of σ in the group $\text{Homeo}(X)$. It is a consequence of our main theorem that σ has trivial centralizer in the group $\text{Homeo}(X)$.

Another way of defining rank-1 transformations is via “cutting and stacking” transformations. In this case, one defines a collection C of

measure-preserving transformations, each obtained by a cutting and stacking construction with intervals. A measure-preserving transformation is said to be rank-1 if it is isomorphic to some element of C . Each $T \in C$ naturally gives rise to a homeomorphism as follows: In the cutting and stacking construction of T , one removes from the interval $[0, 1]$ both the point 0 and the point 1 and each cut-point of the cutting and stacking procedure. By doing this, one removes a countable dense subset from the interval $(0, 1)$, and thus what remains (call it Y) is homeomorphic to Baire space. The transformation T restricted to Y is still a rank-1 measure-preserving transformation, but it is also a homeomorphism of Y . It is a consequence of our main theorem that if $T \in C$ is totally ergodic, then the corresponding homeomorphism has trivial centralizer in the group $\text{Homeo}(Y)$.

The natural topological analogue of ergodicity is this: A homeomorphism f of a Polish space X is *transitive* if for non-empty open sets U and V , there is some $i \in \mathbb{Z}$ so that $f^i(U)$ intersects V . This clearly implies that for each non-empty open set U , the set $\bigcup_{i \in \mathbb{Z}} f^i(U)$ is co-meager in X . A homeomorphism of X is *totally transitive* if f^n is transitive for each $n > 0$. The non-degeneracy condition of the theorem will be satisfied by any homeomorphism that is totally transitive (and some that are not totally transitive).

It should be noted that in [1] there is a definition given for a rank-1 homeomorphism of a Cantor space and several results are proved about homeomorphisms that satisfy that definition. Their analysis is complementary to the analysis of this paper in that their definition for rank-1 is shown in their paper to be equivalent to being conjugate to an odometer, and such homeomorphisms do not satisfy the non-degeneracy condition that our theorem requires.

2. PRELIMINARIES

While the definition given below is for any Polish space X , it should be noted that the existence of a rank-1 homeomorphism of X implies,

by condition (5), that X has a basis of clopen sets, i.e., it is zero-dimensional.

Definition 1. *A homeomorphism f of a Polish space X is rank-1 if there exists a sequence $\{B_n\}$ of strictly decreasing clopen sets and a sequence $\{h_n\}$ of strictly increasing positive integers so that:*

- (1) *the sets $B_n, f(B_n), \dots, f^{h_n-1}(B_n)$ are pairwise disjoint;*
- (2) $\bigcup_{0 \leq i < h_n} f^i(B_n) \subseteq \bigcup_{0 \leq j < h_{n+1}} f^j(B_{n+1});$
- (3) *if $f^j(B_{n+1}) \cap f^i(B_n) \neq \emptyset$ and $0 \leq j < h_{n+1}$ and $0 \leq i < h_n$, then $f^j(B_{n+1}) \subseteq f^i(B_n);$*
- (4) $f^{h_{n+1}-1}(B_{n+1}) \subseteq f^{h_n-1}(B_n);$ *and*
- (5) *the orbits of the sets B_n under f and orbits of the sets $L_n := X \setminus \bigcup_{0 \leq i < h_n} f^i(B_n)$ under f form a basis for the topology of X .*

We will first establish some terminology for rank-1 homeomorphisms. If f is a homeomorphism of a Polish space X , and the sequences $\{B_n\}$ and $\{h_n\}$ witness that f is a rank-1 homeomorphism, then we call the pair $(\{B_n\}, \{h_n\})$ a *tower representation* of f . Every rank-1 homeomorphism has multiple tower representations. For example, if $(\{B_n\}, \{h_n\})$ is a tower representation of f , then by removing a single entry B_k from the sequence $\{B_n\}$ and the corresponding entry h_k from the sequence $\{h_n\}$, one obtains another tower representation of f .

Let f be a rank-1 homeomorphism of a Polish space with a fixed tower representation $(\{B_n\}, \{h_n\})$. For $n \in \mathbb{N}$ and $0 \leq i < h_n$, the set $f^i(B_n)$ is called the i -th level of the stage- n tower. We also call B_n the *base* of the stage- n tower and call $f^{h_n-1}(B_n)$ the *top* of the stage- n tower. We say that h_n is the *height* of the stage- n tower and call L_n the *leftover piece* of the stage- n tower.

The definition of a rank-1 homeomorphism requires that each level of each tower be either a subset of some level of the stage-0 tower or a subset of L_0 . Borrowing terminology from the measure-preserving situation, we sometimes refer to levels of towers that are contained in L_0 as *spacers*. Define $W_n: h_n \rightarrow \{0, 1\}$ so that $W_n(i) = 0$ iff $f^i(B_n)$ is

contained in some level of the stage-0 tower. Since the sequences $\{B_n\}$ and $\{f^{h_n-1}(B_n)\}$ are each decreasing, each W_{n+1} begins and ends with an occurrence of W_n . In particular, each W_{n+1} begins and ends with 0. We can thus define $W_\infty: \mathbb{N} \rightarrow \{0, 1\}$ so that for all $n \in \mathbb{N}$, $W_\infty(i) = W_n(i)$ whenever $W_n(i)$ is defined. There are three possibilities:

- (1) The sequence W_∞ is periodic. In this case, we say that the tower representation of f is *repeating*. One can show that in this case f is not totally transitive.

Odometers form an important class of measure-preserving transformations, and each can be realized as a rank-1 homeomorphism of a Cantor space with a repeating tower representation (in fact, with $L_0 = \emptyset$).

- (2) The sequence W_∞ is not periodic, but there is a bound on the number of consecutive 1s in W_∞ . In this case, we say that the tower representation of f is *non-repeating* and furthermore that it has bounded sequences of consecutive spacers.

The symbolic version of Chacon's (measure-preserving) transformation is a homeomorphism of a Cantor space. The natural choice for a tower representation of this homeomorphism witnesses that it is rank-1 and has bounded sequences of consecutive spacers.

- (3) There are arbitrarily long sequences of consecutive 1s in W_∞ . In this case, we say the tower representation of f is *non-repeating* and furthermore that it has arbitrarily long sequences of consecutive spacers.

Our main result is the following theorem.

Theorem 2.1. *Let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation. If g is a homeomorphism of X that commutes with f , then there is some $i \in \mathbb{Z}$ so that $f^i = g$.*

Before we proceed with the general analysis that will lead us to the proof of the theorem, we prove a technical proposition about the

words W_n that come from a non-repeating tower representation of a rank-1 homeomorphism f . Let $(\{B_n\}, \{h_n\})$ be a non-repeating tower representation of a rank-1 homeomorphism f . Let $m > n$ and consider $B_m, f(B_m), \dots, f^{h_m-1}(B_m)$, the levels of the stage- m tower.

Condition (3) of the definition of rank-1 implies that each level of the stage- m tower is either a subset of a level of the stage- n tower or is contained in L_n . Since $B_m \subseteq B_n$, level i of the stage- m tower is contained in level i of the stage- n tower, for $0 \leq i < h_n$. Thus, W_m begins with an occurrence of W_n .

It is easy to see that either $f^{h_n}(B_m) \subseteq B_n$ or $f^{h_n}(B_m) \subseteq L_n$. Indeed, if $f^{h_n}(B_n) \subseteq f^i(B_n)$ with $0 < i < h_n$, then $f^{h_n-i}(B_m) \subseteq B_n$, which contradicts the fact that $B_n, f(B_n), \dots, f^{h_n-1}(B_n)$ are pairwise disjoint. By similar reasoning, the smallest $j \geq h_n$ for which $f^j(B_m)$ is contained in some level of the stage- n tower is such that $f^j(B_m) \subseteq B_n$. For this j , if $h_n \leq k < j$, then $f^k(B_m) \subseteq L_n$. Thus, the initial W_n in W_m is followed by $(j - h_n)$ -many 1s and then another occurrence of W_n . This pattern continues and W_m can be viewed as a disjoint collection of occurrences of W_n interspersed with 1s. This is described more concretely below.

Let $E_{m,n} = \{i \in [0, h_m) : f^i(B_m) \subseteq B_n\}$. It is clear that if $i \in E_{m,n}$, then W_m has an occurrence of W_n beginning at position i . Such an occurrence of W_n in W_m is called *expected*. The arguments in the preceding paragraph show that each 0 in W_m is a part of exactly one expected occurrence of W_n . In particular, if i and j are consecutive elements of $E_{m,n}$ and $i + h_n \leq k < j$, then $W_m(k) = 1$.

It is possible for W_m to have unexpected occurrences of W_n . For example, if $W_{n+1} = W_n W_n 1 W_n W_n$, for all $n > 0$, then for $0 < n < m$, there is at least one unexpected occurrence of W_n in W_m . However, the following proposition guarantees that knowing a sufficiently large subword of W_m that begins with a specified occurrence of W_n is enough to determine whether that specified occurrence of W_n is expected (and “large enough” is independent of m).

For the proposition below, and the lemma that follows it, recall that we are working with a rank-1 homeomorphism f of a Polish space X that has a non-repeating tower representation $(\{B_n\}, \{h_n\})$.

Proposition 2.2. *For any $n \in \mathbb{N}$, there is some $l(n) \in \mathbb{N}$ so that for any $m > n$, if s is a subword of length $l(n)$ of W_m that begins with an expected occurrence of W_n , then every occurrence of s in W_m begins with an expected occurrence of W_n .*

To prove this proposition, we need the following lemma.

Lemma 2.3. *Suppose W_m has an expected occurrence of W_n that begins at i and is followed by r 1s and then another expected occurrence of W_n . Suppose further that $i < j < i + h_n$ and that W_m also has a (necessarily unexpected) occurrence of W_n that begins at j that is followed by s 1s and then another occurrence of W_n . Then $r = s$.*

Proof. There are four known occurrences of W_n , beginning at i , $i + h_n + r$, j , and $j + h_n + s$. Let α be the subword of W_m beginning at j and ending at $i + h_n - 1$; α has length between 1 and $h_n - 1$, inclusive. Let a be the number of 0s in α (thus, $a > 0$). Let β be the subword of W_m beginning at $i + h_n + r$ and ending at $j + h_n - 1$; β has length between 1 and $h_n - 1 - r$, inclusive. Let b be the number of 0s in β (thus, $b > 0$). Notice that the occurrence of W_n that begins at j consists exactly of $\alpha 1^r \beta$, so there are exactly $a + b$ 0s in W_n . Notice also that the occurrence of W_n that begins at i ends with α and so the word W_n must end with α . Notice also that the occurrence of W_n that begins at $i + h_n + r$ begins with β and so the word W_n must begin with β . By counting the number of 0s in W_n , we see that W_n can also be expressed as $\beta 1^r \alpha$. In particular, the W_n that begins at $i + h_n + r$ has the form $\beta 1^r \alpha$ and also the form $\beta 1^s \alpha'$, where α' is an initial segment of the occurrence of W_n that begins at $j + h_n + s$. Since $\beta 1^r \alpha = \beta 1^s \alpha'$ and both α and α' begin with 0, $r = s$. \square

We now give the proof of Proposition 2.2.

Proof. Suppose $n \in \mathbb{N}$ is such that for each $l \in \mathbb{N}$ there is some $m \in \mathbb{N}$ and a subword s of W_m of length l that begins with an expected occurrence of W_n , so that there is also an occurrence of s in W_m that does not begin with an expected occurrence of W_n .

Since W_∞ is not periodic, there is some $k > n$ so that the number of 1s that separate expected occurrences of W_n in W_k is not constant. Let $l = 2h_k + h_n$. Find $m \in \mathbb{N}$ and s a subword of length l of W_m that begins with an expected occurrence of W_n , and so that W_m has an occurrence of s that does not begin with an expected occurrence of W_n .

First consider the occurrence of s in W_m that begins with an expected occurrence of W_n . Because of the regularity with which expected occurrences of W_n appear in W_m , s can be written as $W_n 1^{r_1} W_n 1^{r_2} \dots W_n 1^{r_t} A$, where A is a proper initial segment (possibly empty) of W_n .

Now consider the occurrence of s in W_m that does not begin with an expected occurrence of W_n (say it begins at position i in W_m). Then there are occurrences of W_n that begin at positions $i, i + h_n + r_1, i + 2h_n + r_1 + r_2, \dots, i + (t-1)h_n + (r_1 + \dots + r_{t-1})$. It is easy to check that since W_n begins and ends with 0, none of these occurrences of W_n are expected in W_m and, moreover, that each of them intersects exactly two expected occurrences of W_n in W_m . Repeated application of Lemma 2.3 shows that $r_1 = r_2 = \dots = r_{t-1}$ and, moreover, that if two occurrences of W_n are completely contained in s and separated only by 1s, then they are separated by exactly r_1 -many 1s.

Also, since there is an expected occurrence of W_n that contains the 0 at position $i + h_n - 1$ and that expected occurrence of W_n ends with 0, we know that $r_1 < h_n$. This clearly implies that in s there is no consecutive sequence of 1s with length h_n .

Any occurrence of s in W_m begins with a 0, which is a part of some expected occurrence of W_k . The length of s is large enough to guarantee that this occurrence of s will completely contain the next expected occurrence of W_k in W_m . This implies that each pair of consecutive

expected occurrences of W_n in W_k are separated by exactly r_1 -many 1s. This contradicts the choice of k . \square

3. INITIAL ANALYSIS

Let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation $(\{B_n\}, \{h_n\})$.

Proposition 3.1. *Each level of the stage- n tower is a disjoint union of at least two levels of the stage- $(n+1)$ tower.*

Proof. Let $0 \leq i < h_n$ and consider $f^i(B_n)$, level i of the stage- n tower. If $x \in f^i(B_n)$, then for some $0 \leq j < h_{n+1}$, $x \in f^j(B_{n+1})$, which implies that $f^j(B_{n+1}) \subseteq f^i(B_n)$. Since the levels of the stage- $(n+1)$ tower are pairwise disjoint, $f^i(B_n)$ is a disjoint union of some levels of the stage- $(n+1)$ tower. To see that $f^i(B_n)$ contains at least two levels of the stage- $(n+1)$ tower, notice that $B_{n+1} \subsetneq B_n$, which guarantees that $f^i(B_{n+1}) \subsetneq f^i(B_n)$. \square

Proposition 3.2. *Every nonempty open set contains a level of a tower.*

Proof. It suffices to show that for each $n \in \mathbb{N}$ and each $i \in \mathbb{Z}$, each of the sets $f^i(B_n)$ and $f^i(L_n)$ contains a level of some tower.

The image, under f , of a non-top level of the stage- m tower is a level of the stage- m tower. The top of the stage- m tower contains at least two levels of the stage- $(m+1)$ tower, and only one of these can be the top of the stage- $(m+1)$. Thus, the image, under f , of the top of the stage- m tower contains a level of the stage- $(m+1)$ tower. Similarly, the pre-image under f of any level of the stage- m tower contains a level of the stage- $(m+1)$ tower. We thus have the following: if A contains a level of a tower, then so does each of $f(A)$ and $f^{-1}(A)$. It now suffices to show that each L_n contains a level of some tower.

If $m > n$, then each level of the stage- m tower is either contained in L_n or is disjoint from L_n . If for every $m > n$, every level of the stage- m tower is disjoint from L_n , then $W_\infty = W_n W_n W_n \dots$, which contradicts

the fact that $(\{B_n\}, \{h_n\})$ is a non-repeating tower representation of f . Therefore, L_n contains some level of some tower. \square

Let $B_\infty = \bigcap B_n$, $T_\infty = \bigcap f^{h_n-1}(B_n)$, and $L_\infty = \bigcap L_n$. We say that a point $x \in X$ is an *interior point* (with respect to f and $(\{B_n, h_n\})$, its tower representation) if no point in the orbit of x is in B_∞ , T_∞ , or L_∞ . Interior points are relatively simple to deal with and will play a crucial role in the proof of the main theorem.

Proposition 3.3. *The set of interior points is comeager in X .*

Proof. It suffices to show that each of the sets L_∞ , B_∞ , and T_∞ are nowhere dense. That L_∞ is nowhere dense is an immediate consequence of the previous proposition. We now prove that B_∞ is nowhere dense. (The proof that T_∞ is nowhere dense is essentially the same.)

Suppose U is a non-empty open set. We need to show that there is some non-empty open $V \subseteq U$ that does not intersect B_∞ . By the previous proposition, U contains a level of a tower. If it is a level that is not the base, then we can set V equal to that level. If, on the other hand, there is a base of a tower, say B_n , contained in U , then we know that B_n is the disjoint union of at least two levels of the stage- $(n+1)$ tower. As only one of those can be the base of the stage- $(n+1)$ tower, another of them can be chosen for V . In any case, we have V , a non-base level of a tower. The clopen set V is disjoint from B_∞ . \square

Proposition 3.4. *Let x be an interior point. Then for each $k \in \mathbb{N}$ there is some $N \in \mathbb{N}$ so that for all $n > N$, there is some i satisfying $k \leq i < h_n - k$ and $x \in f^i(B_n)$.*

Proof. Suppose x is an interior point. Since $x \notin L_\infty$, $x \in f^{j_M}(B_M)$ for some $M \in \mathbb{N}$ and $0 \leq j_M < h_M$. By condition (2) of the definition of rank-1 homeomorphisms, there is, for each $m \geq M$, a unique j_m satisfying $0 \leq j_m < h_m$ so that $x \in f^{j_m}(B_m)$. That the sequences $(j_m : m \geq M)$ and $(h_m - j_m : m \geq M)$ are non-decreasing follows from the fact that $B_{m+1} \subseteq B_m$ and $f^{h_{m+1}-1}(B_{m+1}) \subseteq f^{h_m-1}(B_m)$ for each $m \in \mathbb{N}$.

To prove the proposition, it suffices to show that neither of these sequences is eventually constant. But it is easy to see that if $\{j_m\}$ is eventually constantly c , then $f^{-c}(x) \in B_\infty$, which contradicts the fact that x is an interior point. Similarly, if $\{h_m - j_m\}$ is eventually constantly c , then $f^{c-1}(x) \in T_\infty$, which contradicts the fact that x is an interior point. \square

Corollary 3.5. *Let x be an interior point. Then for each i , there is some n so that $f^i(x)$ is in a level of the stage- n tower that is neither the base nor the top.*

Proposition 3.6. *If x and y are distinct interior points, then there is some level of some tower that contains exactly one of x and y .*

Proof. Suppose that x and y are interior points that are contained in the same levels of the same towers. We must show that $x = y$. We do this by showing that x and y are in the same basic clopen sets.

Suppose that for some $n \in \mathbb{N}$ and some $k \in \mathbb{Z}$, either $f^k(B_n)$ or $f^k(L_n)$ contains exactly one of x and y . By Proposition 3.4 we can find $m > n$ so that there are i, j satisfying $|k| \leq i, j < h_m - |k|$ so that $x \in f^i(B_m)$ and $y \in f^j(B_m)$. But, since x and y are in the same levels of the stage- m tower, $i = j$. Now $f^{-k}(x)$ and $f^{-k}(y)$ are both in the same level of the stage- m tower. By repeated application of condition (3) of the definition of a rank-1 homeomorphism, either there is some level of the stage- n tower that contains both $f^{-k}(x)$ and $f^{-k}(y)$, or $f^{-k}(x)$ and $f^{-k}(y)$ are both in L_n . Thus, neither $f^k(B_n)$ nor $f^k(L_n)$ contains exactly one of x and y .

Therefore, x and y are in the same basic clopen sets and $x = y$. \square

Proposition 3.7. *If U is a nonempty open set and $x \notin \bigcup_{i \in \mathbb{Z}} f^i(U)$, then x is the unique fixed point of f .*

Proof. Let U be nonempty and open. By Proposition 3.2, U contains some level of some tower. So, $\bigcup_{i \in \mathbb{Z}} f^i(U)$ contains some B_n , which contains B_m for every $m > n$. It follows that $\bigcup_{i \in \mathbb{Z}} f^i(U)$ contains

every level of every tower and thus contains everything that is not in L_∞ . In fact, if $x \notin \bigcup_{i \in \mathbb{Z}} f^i(U)$, then the entire orbit of x is contained in L_∞ .

Suppose that $x \notin \bigcup_{i \in \mathbb{Z}} f^i(U)$. Thus, for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}$, both x and $f(x)$ are elements of $f^k(L_n)$ and not elements of $f^k(B_n)$. So, x and $f(x)$ are in the same basic clopen sets and thus, $x = f(x)$.

Now suppose that y is a fixed point of f . Since no element of any level of any tower is a fixed point, $y \in L_\infty$. Thus, for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}$, $y \in f^k(L_n)$ and $y \notin f^k(B_n)$. Thus, x and y are in the same basic clopen sets, and so, $x = y$. \square

4. FURTHER ANALYSIS

4.1. Simplifying Assumptions. As before, let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation $(\{B_n\}, \{h_n\})$. Let g be a homeomorphism of X that commutes with f . We will work towards proving that there is some $k \in \mathbb{Z}$ so that $f^k = g$. For the proof of Lemma 4.5 below, we will distinguish between two cases: either the tower representation $(\{B_n\}, \{h_n\})$ has arbitrarily long sequences of consecutive spacers, or it has bounded sequences of consecutive spacers. If the latter holds, then let a_{max} denote the length of the longest sequence of consecutive spacers in W_∞ . Equivalently, a_{max} is the largest natural number for which there exist $n, m \in \mathbb{N}$ so that $0 \leq m \leq m + a_{max} < h_n$ and so that for each j satisfying $m < j \leq m + a_{max}$, $f^j(B_n) \subseteq L_0$.

Before proceeding further, we make some simplifying assumptions, which we can do without loss of generality. If the tower representation $(\{B_n\}, \{h_n\})$ has bounded sequences of consecutive spacers, we can assume that $h_0 > a_{max}$. Indeed, we can modify the witnessing sequences $\{B_n\}$ and $\{h_n\}$ by deleting the initial entry of each sequence and still have a tower representation for f that is non-repeating and has absolutely bounded sequences of consecutive spacers. Doing this repeatedly

will guarantee that the initial element of the height sequence will be larger than a_{max} .

For the next two simplifying assumptions, consider $g^{-1}(B_0)$. Since g is a homeomorphism, this nonempty set is open and thus must contain $f^m(B_n)$, for some $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Notice that $gf^m(B_n) \subseteq B_0$ and that gf^m is a homeomorphism of X that commutes with f . If gf^m is an integral power of f , then so is g . Thus we may assume that $m = 0$ and thus that $g(B_n) \subseteq B_0$. Also, we may assume that $n = 1$, for otherwise we can delete the entries in the sequences $\{B_n\}$ and $\{h_n\}$ that are indexed by 1 through $(n - 1)$, inclusive. With these two simplifying assumptions we now have the following. If $x \in B_1$, then $g(x) \in B_0$.

4.2. The Set $Z(x)$. For any $x \in X$, let

$$Z(x) = \{i \in \mathbb{Z} : f^i(x) \in B_1\}.$$

The crucial fact is that if x is an interior point, then the set $Z(x)$ contains enough information to recover x .

Proposition 4.1. *If x and y are interior points with $Z(x) = Z(y)$, then $x = y$.*

Proof. Suppose that x and y are distinct interior points and that $Z(x) = Z(y)$. By Proposition 3.6, some level of some tower contains exactly one of x and y . It follows from condition (3) of the definition that for sufficiently large n there is a level of the stage- n tower that contains exactly one of x and y . As x and y are both interior points, each of x and y are in some level of the stage- n tower, for sufficiently large n . Choose such an n and let i and j be such that $0 \leq i, j < h_n$ and $x \in f^i(B_n)$ and $y \in f^j(B_n)$. Without loss of generality, assume $i < j$.

By Proposition 3.4 there is some N so that neither x nor y are in any of the top $l(n)$ levels of the stage- N tower (i.e., for all $0 \leq k < l(n)$, neither $f^k(x)$ nor $f^k(y)$ is in $f^{h_N-1}(B_N)$, the top of the stage- N tower). Recall that $l(n) \in \mathbb{N}$ is such that if s is subword of W_N of length $l(n)$ that begins with an expected occurrence of W_n , then every occurrence

of s in W_N begins with an expected occurrence of W_n (see Proposition 2.2).

Since $Z(x) = Z(y)$, we know that for all $k \in \mathbb{Z}$, $f^k(x) \in B_1$ iff $f^k(y) \in B_1$. It immediately follows from this that if $0 \leq m < h_1$, then for all $k \in \mathbb{Z}$, $f^k(x) \in f^m(B_1)$ iff $f^k(y) \in f^m(B_1)$. In other words, we know that for all k , $f^k(x)$ is in level m of the stage-1 tower iff $f^k(y)$ is in level m of the stage-1 tower. But each level of the stage-1 tower is either completely contained in a level of the stage-0 tower or is disjoint from all levels of the stage-0 tower. Thus for all $k \in \mathbb{Z}$,

$$(1) \quad f^k(x) \in \bigcup_{0 \leq m < h_0} f^m(B_0) \quad \text{iff} \quad f^k(y) \in \bigcup_{0 \leq m < h_0} f^m(B_0).$$

Now consider the points $f^{-i}(x)$, $f^{-i+1}(x)$, \dots , $f^{-i+l(n)-1}(x)$. Since x is not in any of the top $l(n)$ levels of the stage- N tower, these points correspond to a subword of length $l(n)$ in W_N that begins with an expected occurrence of W_n . Call this subword s .

But now consider the points $f^{-i}(y)$, $f^{-i+1}(y)$, \dots , $f^{-i+l(n)-1}(y)$. Since y is not in any of the top $l(n)$ levels of the stage- N tower, these points correspond to a subword of length $l(n)$ in W_N . In fact, by equation (1) above, this subword is exactly s . However, since y is on level j of the stage- n tower and $i < j$, the occurrence of s in W_N that corresponds to the points $f^{-i}(y)$, $f^{-i+1}(y)$, \dots , $f^{-i+l(n)-1}(y)$ does not begin with an expected occurrence of W_n . This is a contradiction. \square

We will analyze the relationship between $Z(x)$ and $Z(g(x))$, but first we mention two facts and prove a proposition. It is easy to see that distinct elements of $Z(x)$ cannot be too close to each other. Indeed, if $f^i(x) \in B_1$, i.e., if $f^i(x)$ is in the base of the stage-1 tower, then $f^{i+1}(x)$ must be in the first level of the stage-1 tower, $f^{i+2}(x)$ must be in the second level of the stage-1 tower, etc. If $f^{i+k}(x)$ is again in the base of the stage-1 tower, with $k > 0$, then it must be the case that $k \geq h_1$. We thus have the following fact.

Fact 1. For any $x \in X$, if $i \neq i'$ are both in $Z(x)$, then $|i - i'| \geq h_1$.

More generally, and for similar reasons, we have:

Fact 2. For any $x \in X$, if $i \neq i'$ and both $f^i(x)$ and $f^{i'}(x)$ are in the same level of the stage- n tower, then $|i - i'| \geq h_n$.

Proposition 4.2. *If x is an interior point, then $Z(x)$ is neither bounded above nor from below.*

Proof. It is clear from the definition of interior point that x is an interior point iff every element of the orbit of x under f is an interior point. It thus suffices to show that for each interior point x , $Z(x)$ contains both a positive and a negative element.

By proposition 3.4 there is some n and some i satisfying $h_1 \leq i < h_n - h_1$ and so that $x \in f^i(B_n)$. Now $f^{-i}(x) \in B_n \subseteq B_1$, so $-i$ is a negative element of $Z(x)$. But also, $f^{h_n-i-1}(x)$ is in the top of the stage- n tower and thus is in the top of the stage-1 tower. So $f^{h_n-i-h_1}(x) \in B_1$ and hence, $h_n - i - h_1$ is a positive element of $Z(x)$. \square

4.3. The Function ϕ_x . We now work towards showing a very rigid connection between $Z(x)$ and $Z(g(x))$, as long as x and $g(x)$ are interior points.

For any $x \in X$, there is a natural way to define a function from $Z(x)$ to $Z(g(x))$. If $i \in Z(x)$, then $f^i(x) \in B_1$. This implies that $gf^i(x) \in B_0$. So there is a unique $m = m(i)$ with $0 \leq m < h_1$ so that $gf^i(x)$ is in level m of the stage-1 tower, i.e., in $f^m(B_1)$. Now $f^{i-m}g(x) \in B_1$, so $i - m \in Z(g(x))$. We thus have a function $\phi_x : Z(x) \rightarrow Z(g(x))$, given by $\phi_x(i) = i - m$. It is clear that for each $i \in Z(x)$,

$$i - h_1 < \phi_x(i) \leq i.$$

A priori, it may be the case that m depends on i . The main line of argument in this paper hinges on the fact, shown in Lemma 4.5 below, that as long as x and $g(x)$ are both interior points, there is no such dependence.

Lemma 4.3. *For any x , the function ϕ_x is an order preserving injection.*

Proof. Suppose i and i' are distinct elements of $Z(x)$ with $i > i'$. By Fact 1 above, we have $i - i' \geq h_1$ and so, $i - h_1 \geq i'$. But we also have $\phi_x(i) > i - h_1$ and $i' \geq \phi_x(i')$. Together these give $\phi_x(i) > \phi_x(i')$. \square

In the next lemma, the levels of the stage- n tower that are subsets of B_1 will be important. For $n > 0$, let r_n denote the number of such levels. Since the definition of rank-1 ensures that the top of the stage- n tower is a subset of the top of the stage-1 tower, the highest level of the stage- n tower that is contained in B_1 is level $h_n - h_1$ (for $n > 0$).

Lemma 4.4. *If x is an interior point, then the function ϕ_x is surjective.*

Proof. Let x be an interior point and suppose $j \in Z(g(x)) \setminus \text{rng}(\phi_x)$. Consider the point $f^{j+h_1-1}(x)$. Since x is interior, we can find some $n > 1$ and $0 \leq m < h_n$ so that $f^{j+h_1-1}(x) \in f^m(B_n)$. Clearly, $f^{j+h_1-1-m}(x)$ is in the base of the stage- n tower.

Now consider the interval $I = [j + h_1 - 1 - m, j + h_n - 1 - m]$. The set $\{f^i(x) : i \in I\}$ contains one element from each of the bottom $h_n - h_1 + 1$ levels of the stage- n tower. This includes all of the levels that are subsets of B_1 . So $|I \cap Z(x)| = r_n$.

Now consider the interval $J = [j - m, j + h_n - 1 - m]$. If $i \in J \cap Z(g(x))$, then $f^i(g(x)) \in B_1$, and so $f^i(g(x))$ is in some level of the stage- n tower that is contained in B_1 . We claim that $|J \cap Z(g(x))| > r_n$. First, if $i \in I \cap Z(x)$, then, since $i - h_1 < \phi_x(i) \leq i$, $\phi_x(i) \in J \cap Z(g(x))$. But j is also in $J \cap Z(g(x))$ and, by assumption, $j \notin \text{rng}(\phi_x)$. Therefore, $|J \cap Z(g(x))| > r_n$.

This implies the existence of distinct $i, i' \in J$ so that both $f^i(g(x))$ and $f^{i'}(g(x))$ are in the same level of the stage- n tower. By Fact 2 above, $|i - i'| \geq h_n$. This is impossible, since $J = [j - m, j + h_n - 1 - m]$. \square

4.4. The Function Ψ_x . For x an interior point, we define a function $\Psi_x : Z(x) \rightarrow \mathbb{N}$ as follows. Let $i \in Z(x)$ and find $j > i$ as small as possible so that $j \in Z(x)$. Let $\Psi_x(i) = j - i$.

Lemma 4.5. *Suppose x and $g(x)$ are interior points. Then for each $i \in Z(x)$, $\Psi_x(i) = \Psi_{g(x)}(\phi_x(i))$.*

The proof of Lemma 4.5 will be done differently for the two cases. We first give the proof in the case that the tower representation $(\{B_n\}, \{h_n\})$ has bounded sequences of consecutive spacers.

Proof. Let x be such that x and $g(x)$ are interior points and let $i \in Z(x)$. We want to show that $\Psi_x(i) = \Psi_{g(x)}(\phi_x(i))$. Let j be the smallest element of $Z(x)$ that is greater than i . We have $\Psi_x(i) = j - i$. Also, since we are in the case with absolutely bounded sequences of consecutive spacers, we have:

$$(2) \quad 0 \leq \Psi_x(i) - h_1 \leq a_{max}$$

Since $\phi_x : Z(x) \rightarrow Z(g(x))$ is an order preserving bijection, we have that $\phi_x(j)$ is the largest element of $Z(g(x))$ that is greater than $\phi_x(i)$. So we have $\Psi_{g(x)}(\phi_x(i)) = \phi_x(j) - \phi_x(i)$. And, as before, we have:

$$(3) \quad 0 \leq \Psi_{g(x)}(\phi_x(i)) - h_1 \leq a_{max}$$

Equations (2) and (3) clearly imply:

$$(4) \quad |\Psi_x(i) - \Psi_{g(x)}(\phi_x(i))| \leq a_{max}$$

We will show that in fact, $\Psi_x(i) = \Psi_{g(x)}(\phi_x(i))$.

First, consider the point $f^i g(x)$. Since $i \in Z(x)$, $f^i g(x) \in B_0$. We claim that also $f^{i+\Psi_{g(x)}(\phi_x(i))} g(x) \in B_0$. Indeed, since $f^i g(x) \in B_0$, the point $f^i g(x)$ must be in some level of the stage-1 tower. Let $0 \leq m < h_1$ be such that $f^i g(x) \in f^m(B_1)$. Then $f^{i-m} g(x) \in B_1$ and $\phi_x(i) = i - m$. Now $f^{i-m+\Psi_{g(x)}(\phi_x(i))} g(x) \in B_1$, and so $f^{i+\Psi_{g(x)}(\phi_x(i))} g(x) \in f^m(B_1)$. Since we know that $f^m(B_1)$ intersects B_0 , it must be contained in B_0 . Thus $f^{i+\Psi_{g(x)}(\phi_x(i))} g(x) \in B_0$.

Next, consider the point $f^j g(x)$. Since $j \in Z(x)$, $f^j g(x) \in B_0$. But $j = i + \Psi_x(i)$, so $f^{i+\Psi_x(i)} g(x) \in B_0$.

Suppose that $\Psi_{g(x)}(\phi_x(i)) \neq \Psi_x(i)$. Then $i + \Psi_{g(x)}(\phi_x(i)) \neq i + \Psi_x(i)$. Fact 2 then implies that

$$|(i + \Psi_{g(x)}(\phi_x(i))) - (i + \Psi_x(i))| \geq h_0.$$

Since $h_0 > a_{max}$, we have

$$|\Psi_{g(x)}(\phi_x(i)) - \Psi_x(i)| > a_{max},$$

which contradicts equation (4) above. \square

The proof of Lemma 4.5 is more involved in the case that the tower representation $(\{B_n\}, \{h_n\})$ has arbitrarily long sequences of spacers. In this case, we need to show that Ψ_x and $\Psi_{g(x)}$ exhibit an almost periodic behavior. Since the elements of $Z(x)$ are not equally spaced, it will be easier to describe this type of periodicity if we first identify \mathbb{Z} with $Z(x)$. This can be done because, by Proposition 4.2, $Z(x)$ is neither bounded from above nor from below.

Fix some $i_0 \in Z(x)$ and let this correspond to $0 \in \mathbb{Z}$. This extends uniquely to an order preserving correspondence of \mathbb{Z} with $Z(x)$. Let i_k denote the element of $Z(x)$ that corresponds to $k \in \mathbb{Z}$. This correspondence between \mathbb{Z} and $Z(x)$ extends through the order preserving bijection $\phi : Z(x) \rightarrow Z(g(x))$ to a correspondence between \mathbb{Z} and $Z(g(x))$. For $k \in \mathbb{Z}$, let j_k denote $\phi(i_k) \in Z(g(x))$. In the ensuing discussion of Ψ_x and $\Psi_{g(x)}$ we will take the domain of each function to be \mathbb{Z} ; that is, we will write $\Psi_x(k)$ in place of $\Psi_x(i_k)$ and we will write $\Psi_{g(x)}(k)$ in place of $\Psi_{g(x)}(j_k)$.

Recall that for $n > 0$, r_n is the number of levels in the stage- n tower that are contained in B_1 .

Claim 1. Let x be an interior point.

- (1) For each $n > 1$, Ψ_x is constant each congruence class mod r_n except one. On this last congruence class mod r_n , Ψ_x is unbounded.
- (2) For each $k \in \mathbb{Z}$, there is an $n > 1$ so that Ψ_x is constant on the congruence class of k mod r_n .

Proof. For each $n > 0$ we will define R_n , a sequence of natural numbers of length $r_n - 1$. Let z be any point in the base of the stage- n tower. Let $\{j_0, j_1, \dots, j_{r_n-1}\}$ enumerate, from smallest to largest, the elements of the set $\{j \in [0, h_n - 1] : f^j(z) \in B_1\}$. We now define R_n to be the sequence $(j_1 - j_0, j_2 - j_1, \dots, j_{r_n-1} - j_{r_n-2})$.

It is clear that the definition of R_n is independent of which point $z \in B_n$ is chosen. So whenever $i_k \in Z(x)$ is such that $f^{i_k}(x)$ is in the base of the stage- n tower, there is an occurrence of the word R_n that begins at position k in Ψ_x . But since x is an interior point, if k is such that $f^{i_k}(x) \in B_n$, then both $f^{i_k+r_n}(x)$ and $f^{i_k-r_n}(x)$ are in the base of the stage- n tower. So for such a k the function Ψ_x is constant on the congruence class of $k + m \bmod r_n$, for each $0 \leq m < r_n - 1$. But since we are in the case that the tower representation has unbounded sequences of consecutive spacers, the last congruence class mod r_n must be unbounded. This proves part 1.

Let $k \in \mathbb{Z}$. Since $f^{i_k}(x)$ is in the base of the stage-1 tower, $f^{i_k+h_1-1}(x)$ is in the top of the stage-1 tower. Since x is an interior point, there is some n so that $f^{i_k+h_1-1}(x)$ is not in the top of the stage- n tower. For this n , Ψ_x is constant on the congruence class of $k \bmod r_n$. \square

We now give the proof of Lemma 4.5 in the case that the tower representation $(\{B_n\}\{h_n\})$ has arbitrarily long sequences of spacers.

Proof. Let x and $g(x)$ be interior points. We want to show that $\Psi_x = \Psi_{g(x)}$. Suppose that $\Psi_x(i) \neq \Psi_{g(x)}(i)$. Choose n so that $\Psi_{g(x)}$ is constant on the set $\{i + k(r_n) : k \in \mathbb{Z}\}$ and let j be such that $\Psi_{g(x)}$ is unbounded on the set $\{j + k(r_n) : k \in \mathbb{Z}\}$. Clearly i and j are in different congruence classes mod r_n . If Ψ_x is unbounded on the set $\{j + k(r_n) : k \in \mathbb{Z}\}$, then Ψ_x agrees with $\Psi_{g(x)}$ at each position except perhaps those in $\{j + r(z_n) : r \in \mathbb{Z}\}$ (which contradicts the fact that Ψ_x and $\Psi_{g(x)}$ differ at i). Thus Ψ_x is constant on the set $\{j + k(r_n) : k \in \mathbb{Z}\}$.

Now find $k \in \mathbb{Z}$ so that $\Psi_{g(x)}(k) - \Psi_x(k) \geq h_1$. We have that $i_{k+1} - i_k = \Psi_x(k)$ and also that $j_{k+1} - j_k = \Psi_{g(x)}(k)$. It follows that

$$\Psi_{g(x)}(k) - \Psi_x(k) = (j_{k+1} - i_{k+1}) + (i_k - j_k).$$

But we know that $j_{k+1} - i_{k+1} \leq 0$ and that $i_k - j_k < h_1$. Thus, $\Psi_{g(x)}(k) - \Psi_x(k) < h_1$, a contradiction. \square

Lemma 4.6. *If x and $g(x)$ are interior points, then for some $k \in \mathbb{Z}$, $f^k(x) = g(x)$.*

Proof. Suppose x and $g(x)$ are interior points. We know that for each $i \in Z(x)$, $i - h_1 < \phi_x(i) \leq i$.

We claim that $i - \phi_x(i)$ is independent of i . Indeed, if $i - \phi_x(i)$ is not independent of i , then we can find consecutive i and j in $Z(x)$ so that $i - \phi_x(i) \neq j - \phi_x(j)$. By saying that i and j are consecutive elements of $Z(x)$, we formally mean that $j \in Z(x)$ is as small as possible satisfying $j > i$. So we have

$$\phi_x(j) - \phi_x(i) \neq j - i.$$

Since i and j are consecutive elements of $Z(x)$, $j - i = \Psi_x(i)$. Since $\phi_x : Z(x) \rightarrow Z(g(x))$ is an order preserving bijection we also have that $\phi_x(i)$ and $\phi_x(j)$ are consecutive elements of $Z(g(x))$ and so $\phi_x(j) - \phi_x(i) = \Psi_{g(x)}(\phi_x(i))$. But then

$$\Psi_{g(x)}(\phi_x(i)) \neq \Psi_x(i)$$

and this contradicts Lemma 4.5.

Thus, $i - \phi_x(i)$ is independent of i . Let k be such that for all $i \in Z(x)$, $i - \phi_x(i) = k$. Since $\phi_x : Z(x) \rightarrow Z(g(x))$ is a bijection we have that $i \in Z(x)$ iff $i - k \in Z(g(x))$. But clearly, $i \in Z(x)$ iff $i - k \in Z(f^k(x))$. Thus, $Z(f^k(x)) = Z(g(x))$ and, by Proposition 4.1, $f^k(x) = g(x)$. \square

5. PROOF OF THE MAIN THEOREM

We now prove Theorem 2.1.

Proof. Let f be a rank-1 homeomorphism of a Polish space X with a non-repeating tower representation and let g be a homeomorphism of X that commutes with f . By Proposition 3.3, the set of interior points is comeager in X . So the set $\{x \in X : x \text{ and } g(x) \text{ are interior points}\}$ is also comeager in X .

Since both g and f are homeomorphisms of X , each $A_i = \{x \in X : g(x) = f^i(x)\}$ is closed. By Lemma 4.6, each element of the comeager set $\{x \in X : x \text{ and } g(x) \text{ are interior points}\}$ is in some A_i . Therefore, some A_i is dense in some nonempty open set U . Since A_i is closed, it contains U . Since g and f commute, A_i is invariant under T . By Proposition 3.7, $\bigcup_{i \in \mathbb{Z}} f^i(U)$ is either all of X or all of X except the unique fixed point of f . But the unique fixed point of f must be a fixed point of g (since g and f commute) and thus must be in A_i .

We now have that A_i is all of X . Thus $g = f^i$. \square

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